

Gaussian Effective Potential Analysis of Sinh(Sine)-Gordon Models by New Regularization-Renormalization Scheme

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Abstract

Using the new regularization and renormalization scheme recently proposed by Yang and used by Ni et al , we analyse the sine-Gordon and sinh-Gordon models within the framework of Gaussian effective potential in $D + 1$ dimensions. Our analysis suffers no divergence and so does not suffer from the manipulative obscurities in the conventional analysis of divergent integrals. Our main conclusions agree exactly with those of Ingermanson for $D = 1, 2$ but disagree for $D = 3$: the $D = 3$ sinh(sine)-Gordon model is non-trivial. Furthermore, our analysis shows that for $D = 1, 2$, the running coupling constant (RCC) has poles for sine-Gordon model ($\gamma^2 < 0$) and the sinh-Gordon model ($\gamma^2 > 0$) has a possible critical point γ_c^2 while for $D = 3$, the RCC has poles for both $\gamma^2 > 0$ and $\gamma^2 < 0$.

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1 Introduction

The "Gaussian effective potential" (GEP) has proven to be a powerful non-perturbative approach in quantum field theories (QFT). Using the GEP approach, Stevenson etc. found two distinct, trivial versions of the 3+1 dimensional $\lambda\phi^4$ theory: the "precarious ϕ^4 theory" and the "autonomous ϕ^4 theory" [1], and thus provided a new view point about the triviality of $\lambda\phi^4$ model as a physical theory. Also by GEP, Ingermanson examined the generalized sinh-Gordon and sine-Gordon models in $D + 1$ dimensions [2]. The Lagrangian for the model takes in general the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{\gamma^2}[\text{ch}(\gamma\phi) - 1]$$

where m and γ are the mass and coupling constant respectively at tree level. If $\gamma^2 > 0$, the classical potential is a cosh curve with a single minimum at the origin; if $\gamma^2 < 0$, it is actually a sine-Gordon model with an infinite number of degenerate minimum of the potential. The limiting case $\gamma^2 \rightarrow 0$ is usually understood to be a free theory of masses m . When $D = 1$, the sine-Gordon model is equivalent to a group of other models [3], namely, the massive Thirring model [4], the Coupled Gas [5], the continuum limit of the xyz spin- $\frac{1}{2}$ model [6] and the massive $O(2)$ non-linear σ -model. It is convenient to define $\beta^2 = -\gamma^2$ for discussing the sine-Gordon model. It has been shown that the $D = 1$ sine-Gordon model is superrenormalizable for $0 \leq \beta^2 \leq 4\pi$; renormalizable for $4\pi \leq \beta^2 \leq 8\pi$, and nonrenormalizable for $\beta^2 > 8\pi$ [7], the last property was first discovered by Coleman [4]. Based on GEP, Ingermanson concluded that for $D \geq 3$, the model (1) can exist only as a free theory while for $D < 3$, the vacuum is unstable over a certain range of the coupling constant. In Ingermanson's analysis, the integrals

$$I_n^D(\mu^2) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{\sqrt{\mathbf{p}^2 + \mu^2}}{(\mathbf{p}^2 + \mu^2)^n}$$

may be divergent or finite. The divergent ones were dealt with without using any cutoff procedure or regularization procedure and were just taken to be as though finite most of the time, and the whole analysis seems to be regularization scheme independent. Yet for $D \geq 3$, the fact that I_2^D is divergent was used to lead to the conclusion that the interacting theory is inconsistent for $D \geq 3$. Hence, the rule that taking I_n^D as finite was violated here and there exists such a kind of mathematical obscurity.

To eliminate this obscurity, we intend to re-analyse the model (1) by the new regularization-renormalization (R-R) scheme, which was proposed by Yang [8] and used by Ni et al recently [9].

though the "derivative regularization" trick has been evolving in the literatures for many years [18]. The spirit is like this: when encountering a superficially divergent Feynmann diagram into FDI, we first differentiate it with respect to some parameter such as a mass parameter enough until it becomes convergent and the integration can be done. Then we reintegrate it with respect to the same parameter the same times. The result is to be taken as the definition of the original FDI. Then instead of divergence, some arbitrary constants appear in FDI. The appearance of arbitrary constants indicates some lack of theoretical knowledge about the model at QFT level under consideration. The determination of them is beyond the ability of the QFT, instead, they should be fixed by experiment via some suitable renormalization procedure. This new R-R scheme has turned out to be successful in that the whole analysis is quite clearcut and it can give a prediction of Higgs mass, $m_H = 138$ GeV in the standard model[11]. Also it provides an elegant calculation in QED, e.g. Lamb shift [12]. In this paper our main conclusions agree exactly with those of Ingermarsen for $D = 1, 2$. But for $D = 3$ there is an important discrepancy : the $D = 3$ sinh(sine)-Gordon model may be non-trivial. Furthermore, our analysis shows that for $D = 1, 2$, the running coupling constant (RCC) has poles for $\gamma^2 < 0$ and the sinh-Gordon model has a possible critical point γ_c^2 for $D = 3$, the RCC has poles for both $\gamma^2 > 0$ and $\gamma^2 < 0$. In section 2, we give a general analysis of the model (1) in the Schrödinger representation and present some known results. In section 3, we analyse the model for $D = 1, 2, 3$ respectively by the new R-R scheme. The last section is devoted to discussions.

2 General Analysis

2.1 GEP and running coupling constant(RCC)

The Lagrangian (1) can be rewritten as

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$V(\phi) = \frac{1}{2}(\partial_i\phi)^2 + \frac{m^2}{\gamma^2}(\cosh\gamma\phi - 1)$$

The canonical momentum conjugate to ϕ is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$

and the Hamiltonian reads

$$H = \int d^D\mathbf{x}[\frac{1}{2}\pi^2 + V(\phi)]$$

Effective Potential, Sine-Gordon

The quantization is realized through

$$[\pi(x_0, \mathbf{x}), \phi(x_0, \mathbf{y})] = -i\delta^D(\mathbf{x} - \mathbf{y})$$

which can be satisfied if

$$\pi = -i\frac{\delta}{\delta\phi} + G(\phi)$$

In particular, we often choose $G(\phi) = 0$. In Schrödinger representation, the state is described by a wave functional $\Psi[\phi]$ which satisfies the Schrödinger equation

$$H\Psi[\phi] = E\Psi[\phi]$$

The first step in Gaussian variational method is to make an ansatz for the Schrödinger wave functional for the vacuum

$$\Psi[\phi, \Phi, \mathcal{P}, f] = \mathcal{N}_f \exp\left\{i \int \mathcal{P}_{\mathbf{x}} \phi_{\mathbf{x}} - \frac{1}{2} \int_{\mathbf{x}, \mathbf{y}} (\phi_{\mathbf{x}} - \Phi_{\mathbf{x}}) f_{\mathbf{x}, \mathbf{y}} (\phi_{\mathbf{y}} - \Phi_{\mathbf{y}})\right\}$$

The \mathcal{P}, Φ, f are variational parameters. The energy of the variational state eq.(10) is

$$E[\Phi, \mathcal{P}, f] \equiv \langle \Psi | H | \Psi \rangle = \int_{\mathbf{x}} \left\{ \frac{1}{2} \mathcal{P}_{\mathbf{x}}^2 + \frac{1}{2} (\partial_i \Phi)^2 + \frac{m^2}{\gamma^2} [Z_{\mathbf{x}} \text{ch} \gamma \Phi - 1] + \frac{1}{4} [f_{\mathbf{xx}} - \int_{\mathbf{y}} \delta_{\mathbf{xy}} \nabla_{\mathbf{x}}^2 f_{\mathbf{xy}}^{-1}] \right\}$$

where

$$Z_{\mathbf{x}} \equiv \exp\left(\frac{1}{4} \gamma^2 f_{\mathbf{xx}}^{-1}\right)$$

We are interested in finding the effective potential, so we consider the energy of the state with constant classical field $\Phi, \partial_i \Phi = 0$. The extremum energy configuration clearly satisfies the constraint $\mathcal{P} = 0$. The variational equation

$$\frac{\delta E}{\delta f_{\mathbf{xy}}} = 0$$

gives the general forms of $f_{\mathbf{xy}}$ and $f_{\mathbf{xy}}^{-1}$ as

$$f_{\mathbf{xy}} = \int \frac{d^D \mathbf{p}}{(2\pi)^D} \sqrt{\mathbf{p}^2 + \mu^2} \cos \mathbf{p} \cdot (\mathbf{x} - \mathbf{y})$$

$$f_{\mathbf{xy}}^{-1} = \int \frac{d^D \mathbf{p}}{(2\pi)^D} \frac{\cos \mathbf{p} \cdot (\mathbf{x} - \mathbf{y})}{\sqrt{\mathbf{p}^2 + \mu^2}}$$

Using $I_n^D(\mu^2)$ in eq.(2), we have (we often omit the superscript D)

$$f_{\mathbf{xx}} = I_0(\mu^2), \quad f_{\mathbf{xx}}^{-1} = I_1(\mu^2) = 2 \frac{\partial}{\partial \mu^2} I_0$$

Effective Potential, Sine-Gordon

The energy density \mathcal{E} is a function of Φ and μ^2

$$\mathcal{E}(\Phi, \mu^2) = \frac{1}{2}I_0(\mu^2) - \frac{1}{4}\mu^2 I_1(\mu^2) + \frac{m^2}{\gamma^2}[Z_m(\mu^2)\text{ch}\gamma\Phi - 1]$$

where

$$Z_m(\mu^2) = \exp\left[\frac{1}{4}\gamma^2 I_1(\mu^2)\right]$$

According to Ritz variational principle[19], any stationary state (10) is an eigenstate of the discrete spectrum of H , and the corresponding eigenvalue is the stationary value of the function (17).

we consider the stationary points $(\bar{\mu}^2, \bar{\Phi})$ for \mathcal{E} which are solutions of the equations

$$\frac{\partial \mathcal{E}}{\partial \Phi} = \frac{m^2}{\gamma} Z_m(\mu^2) \text{sh}\gamma\Phi = 0$$

$$\frac{\partial \mathcal{E}}{\partial \mu^2} = \frac{1}{8}I_2(\mu^2)[\mu^2 - m^2 Z_m(\mu^2)\text{ch}\gamma\Phi] = 0$$

i.e

$$\text{sh}\gamma\bar{\Phi} = 0$$

$$\bar{\mu}^2 = m^2 Z_m(\bar{\mu}^2) \text{ch}\gamma\bar{\Phi}$$

(As one is interested in the effective potential, one may consider the stationary point $\bar{\mu}^2$ and $\bar{\Phi}$ free as we will do in the following.) Clearly, if $\gamma^2 > 0$, $\bar{\mu}^2$ is always positive and we have only solution $(\bar{\mu}^2, \bar{\Phi} = 0)$. Instead, if $\gamma^2 = -\beta^2 < 0$, $\bar{\mu}^2$ is positive only when $\cos\beta\bar{\Phi} > 0$, so necessary that $(2n - \frac{1}{2})\pi \leq \beta\bar{\Phi} \leq (2n + \frac{1}{2})\pi$, ($n \in \mathbb{N}$). but eq.(21) confines it to be $\sin\beta\bar{\Phi} = 0$ we have an infinite number of stationary points $(\bar{\mu}^2, \bar{\Phi}_n = 2n\pi)$. It is evident that for all stationary points, the energy takes the same value. Therefore, for negative γ^2 , the stationary states are infinitely degenerate.

To guarantee that the stationary point is an local minimum, we have to demand that the matrix

$$\mathcal{M} = \begin{pmatrix} \frac{\partial^2 \mathcal{E}}{\partial \Phi^2} & \frac{\partial^2 \mathcal{E}}{\partial \Phi \partial (\mu^2)} \\ \frac{\partial^2 \mathcal{E}}{\partial (\mu^2) \partial \Phi} & \frac{\partial^2 \mathcal{E}}{\partial (\mu^2)^2} \end{pmatrix}$$

is positively definite. Since

$$\frac{\partial^2 \mathcal{E}}{\partial \Phi \partial (\mu^2)} = -\frac{1}{8}\gamma m^2 I_2 Z_m(\mu^2) \text{sh}\gamma\Phi$$

$$\frac{\partial^2 \mathcal{E}}{\partial (\mu^2)^2} = \frac{1}{8}I_2 - \frac{3}{16}\mu^2 I_3 + \frac{3}{16}m^2 I_3 Z_m(\mu^2) \text{ch}\gamma\Phi + \frac{1}{64}\gamma^2 m^2 I_2^2 Z_m(\mu^2) \text{ch}\gamma\Phi$$

$$\frac{\partial^2 \mathcal{E}}{\partial \Phi^2} = m^2 Z_m(\mu^2) \text{ch}\gamma\Phi$$

we have from (21) and (22)

$$\frac{\partial^2 \mathcal{E}}{\partial(\mu^2)^2}|_{\bar{\mu}^2, \bar{\Phi}} = \frac{1}{8}I_2(1 + \frac{1}{8}\bar{\mu}^2\gamma^2 I_2) \quad \frac{\partial^2 \mathcal{E}}{\partial\phi^2}|_{\bar{\mu}^2, \bar{\Phi}} = \bar{\mu}^2, \quad \frac{\partial^2 \mathcal{E}}{\partial\Phi\partial(\mu^2)}|_{\bar{\Phi}, \bar{\mu}^2} = 0$$

So for \mathcal{M} to be positively definite, we should have

$$\frac{1}{8}I_2(1 + \frac{1}{8}\bar{\mu}^2\gamma^2 I_2) > 0$$

The GEP is defined as

$$V_G(\Phi) = \mathcal{E}(\Phi, \mu^2(\Phi))$$

where the functional relation of μ^2 to Φ is the same as (22) of $\bar{\mu}^2$ to $\bar{\Phi}$. Like the usual effective potential V_{eff} obtained by loop expansions[20], V_G has also the physical interpretation: it is the minimum of the expectation value of the energy density for all states constrained by the condition that the ϕ has expectation value Φ . Using (22), V_G can be written as

$$V_G = \frac{1}{2}I_0(\mu^2) - \frac{1}{4}\mu^2 I_1(\mu^2) + \frac{\mu^2 - m^2}{\gamma^2}$$

It is straight forward to check that

$$\begin{aligned} \frac{dV_G}{d(\mu^2)} &= \frac{1}{\gamma^2}(1 + \frac{1}{8}\mu^2\gamma^2 I_2(\mu^2)) \\ \frac{d\mu^2}{d\Phi} &= \gamma\mu^2 \text{th}\gamma\Phi [1 + \frac{1}{8}\mu^2\gamma^2 I_2(\mu^2)]^{-1} \end{aligned}$$

and so

$$\frac{dV_G}{d\Phi} = \frac{\mu^2}{\gamma} \text{th}\gamma\Phi$$

Clearly, V_G acquires its minimum at $\Phi_0 = 0$, which agrees with $\bar{\Phi}$. (In general, the stationary points of an arbitrary function $f(x, y)$ agree with those of $f(x(y), y)$, where x as a function of y is determined by $\partial f / \partial x = 0$, but whether $f(x, y)$ and $f(x(y), y)$ acquire their maximum or minimum simultaneously just depends.)

For later use, we calculate the following derivatives. First we have

$$\frac{d^2 V_G}{d\Phi^2} = \frac{1}{\gamma} \frac{d\mu^2}{d\Phi} \text{th}\gamma\Phi + \frac{\mu^2}{\gamma} \frac{d\text{th}\gamma\Phi}{d\Phi}$$

From (32) and $d\text{th}\gamma\Phi/d\Phi = \gamma/\text{ch}^2\gamma\Phi$, we have

$$\frac{d^2 V_G}{d\Phi^2} = \mu^2 \text{th}^2\gamma\Phi (1 + \frac{1}{8}\mu^2\gamma^2 I_2)^{-1} + \mu^2 \text{ch}^{-2}\gamma\Phi$$

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$$\begin{aligned}
\frac{d^3 V_G}{d\Phi^3} &= \gamma \mu^2 \text{th}^3 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-2} + 3 \gamma \mu^2 \frac{\text{th} \gamma \Phi}{\text{ch}^2 \gamma \Phi} (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-1} - \\
&\gamma (\mu^2)^2 \text{th}^3 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-3} (\frac{1}{8} \gamma^2 I_2 - \frac{3}{16} \mu^2 \gamma^2 I_3) - 2 \gamma \mu^2 \text{ch}^{-3} \gamma \Phi \text{sh} \gamma \Phi \\
\frac{d^4 V_G}{d\Phi^4} &= \gamma^2 \mu^2 \text{th}^4 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-3} + 6 \gamma^2 \mu^2 \frac{\text{th}^2 \gamma \Phi}{\text{ch}^2 \gamma \Phi} (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-2} \\
&- 4 (\gamma \mu^2)^2 \text{th}^4 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-4} (\frac{1}{8} \gamma^2 I_2 - \frac{3}{16} \mu^2 \gamma^2 I_3) + 3 \gamma^2 \mu^2 \frac{1 - 2 \text{sh}^2 \gamma \Phi}{\text{ch}^4 \gamma \Phi} (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-1} \\
&- 6 (\gamma \mu^2)^2 \frac{\text{th}^2 \gamma \Phi}{\text{ch}^2 \gamma \Phi} (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-3} (\frac{1}{8} \gamma^2 I_2 - \frac{3}{16} \mu^2 \gamma^2 I_3) \\
&+ 3 \gamma^2 (\mu^2)^3 \text{th}^4 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-5} (\frac{1}{8} \gamma^2 I_2 - \frac{3}{16} \mu^2 \gamma^2 I_3)^2 \\
&- \gamma^2 (\mu^2) \text{th}^4 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-4} (-\frac{3}{8} \gamma^2 I_3 + \frac{5}{32} \mu^2 \gamma^2 I_4) \\
&- 2 \gamma^2 \mu^2 \frac{\text{th}^2 \gamma \Phi}{\text{ch}^2 \gamma \Phi} (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-1} - 2 \gamma^2 \mu^2 (\text{ch}^{-2} \gamma \Phi - \frac{3 \text{th}^2 \gamma \Phi}{\text{ch}^2 \gamma \Phi})
\end{aligned}$$

At Φ_0 , $\text{ch} \gamma \Phi_0 = 0$, we have

$$\frac{d^4 V_G}{d\Phi^4} \Big|_{\Phi_0} = 3 \gamma^2 \mu^2 (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-1} - 2 \gamma^2 \mu^2$$

The renormalization is carried out at Φ_0 (it will be referred to as Φ_0 -renormalization) and renormalized mass and coupling constant are defined by

$$m_R^2 \equiv \frac{d^2 V_G}{d\Phi^2} \Big|_{\Phi_0}$$

$$m_R^2 \gamma_R^2 \equiv \frac{d^4 V_G}{d\Phi^4} \Big|_{\Phi_0}$$

We see from (40) that the renormalization of the coupling constant depends on that of the mass. We deduce from (35) and (38) that

$$m_R^2 = \mu^2$$

$$m_R^2 \gamma_R^2 = 3 \gamma^2 \mu^2 (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-1} - 2 \gamma^2 \mu^2$$

. Eq(41) just asserts that the renormalized mass, which is in general the energy difference of particle state and the vacuum [21], equals the variational parameter.

2.2 The Running Coupling Constant

Analogous to that in the $\lambda\phi^4$ model[11], the running coupling constant (RCC) is defined

$$\begin{aligned}
 \gamma^2[\mu^2(\Phi)] &= \frac{d^4 V_G}{d\Phi^4} / \frac{d^2 V_G}{d\Phi^2} \\
 &= \gamma^2 \text{th}^4 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-2} (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1} \\
 &\quad + 6 \gamma^2 \text{th}^2 \gamma \Phi \text{ch}^{-2} \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-1} (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1} \\
 &\quad - 4 \gamma^2 \mu^2 \text{th}^4 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-3} (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1} (\frac{1}{8} \gamma^2 I_2 - \frac{3}{16} \mu^2 \gamma^2 I_3) \\
 &\quad + 3 \gamma^2 (1 - 2 \text{sh}^2 \gamma \Phi) \text{ch}^{-4} \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1} \\
 &\quad - 6 \gamma^2 \mu^2 \text{th}^2 \gamma \Phi \text{ch}^{-2} \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-2} (\frac{1}{8} \gamma^2 I_2 - \frac{3}{16} \mu^2 \gamma^2 I_3) (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1} \\
 &\quad + 3 \gamma^2 (\mu^2)^2 \text{th}^4 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-4} (\frac{1}{8} \gamma^2 I_2 - \frac{3}{16} \mu^2 \gamma^2 I_3)^2 (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1} \\
 &\quad - \gamma^2 (\mu^2)^2 \text{th}^4 \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2)^{-3} (-\frac{3}{8} \gamma^2 I_3 + \frac{5}{32} \mu^2 \gamma^2 I_4) (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1} \\
 &\quad - 2 \gamma^2 \text{th}^2 \gamma \Phi \text{ch}^{-2} \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1} \\
 &\quad - 2 \gamma^2 (1 - 3 \text{th}^2 \gamma \Phi) \text{ch}^{-2} \gamma \Phi (1 + \frac{1}{8} \mu^2 \gamma^2 I_2) (1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi)^{-1}
 \end{aligned}$$

It can be easily seen that $\gamma^2[\mu^2(\Phi)]$ has poles at

$$\text{ch} \gamma \Phi = 0$$

$$1 + \frac{1}{8} \mu^2 \gamma^2 I_2 = 0$$

and

$$1 + \frac{1}{8} \mu^2 \gamma^2 I_2 \text{ch}^{-2} \gamma \Phi = 0$$

The poles corresponding to eqs(44)-(46) are of the fourth, first and the fourth order respective

3 The New R-R Analysis

3.1 The $D = 1$ Case

Following the spirit of the new regularization, we have

$$I_2 = \frac{1}{\pi \mu^2} \quad I_1 = -\frac{1}{2\pi} \ln \frac{\mu^2}{\mu_s^2} \quad I_0 = C - \frac{\mu^2}{4\pi} (\ln \frac{\mu^2}{\mu_s^2} - 1)$$

where C, μ_s^2 are two arbitrary constants. It can be easily seen that only μ_s^2 is non-trivial and is determined by some renormalization scheme. Thus we only need the mass renormalization condition. We choose such a scheme that *the Φ_0 -renormalized mass is just the mass given at the tree level*

$$m_R^2 = m^2$$

So from (22) and (41) we have $Z_m = 1$ which fixes $I_1 = 0$, thus $\mu_s^2 = m^2$. Consequently renormalized coupling constant γ_R^2 is

$$\gamma_R^2 = \frac{1 - \frac{1}{4\pi}\gamma^2}{1 + \frac{1}{8\pi}\gamma^2}\gamma^2$$

i.e. the coupling constant endures a finite renormalization which can provide us with some important information about the model after quantization. Since it is usually expected that quantum corrections are small so γ_R^2 and γ^2 should be of the same sign, we should have that

$$\frac{1 - \frac{1}{4\pi}\gamma^2}{1 + \frac{1}{8\pi}\gamma^2} > 0$$

which implies that

$$-8\pi < \gamma^2 < 4\pi$$

On the other hand, the optimal $\bar{\Phi}$ for $\mathcal{E}(\mu^2, \Phi)$ incidentally coincides with the minimum Φ_0 for \mathcal{E} . From (28) we have

$$\gamma^2 > -8\pi$$

So the two conditions agree well and confirm that there exists a critical value for γ^2 , i.e. for $-8\pi < \gamma^2 < 4\pi$, but for $\gamma^2 = -\beta^2 < 0, \beta_c^2 = 8\pi$. It seems that for sinh-Gordon model, $\gamma^2 = 4\pi$ is a critical point at which $\gamma_R^2 = 0$, but whether the higher vertices also become zero, i.e. whether the model becomes a free one has to be confirmed by further analysis.

Consider now the low-lying excited states relative to $\Psi[\mu^2(\Phi), \Phi]$. The gap equation (22) turns out to be

$$\left(\frac{\mu^2}{m^2}\right)^{1+\frac{\gamma^2}{8\pi}} = \text{ch}\gamma\Phi$$

Since $I_1 = -\frac{1}{2\pi} \ln \frac{\mu^2(\Phi)}{m^2}$, we must have $\mu^2(\Phi) \geq m^2$ for $\gamma^2 > 0$ and $\mu^2 \leq m^2$ for $0 \leq -\gamma^2 < 8\pi$. The mass parameter at tree level provides a lower bound for the particle mass of low-lying excited states if $\gamma^2 > 0$ while an upper bound if $\gamma^2 < 0$ after the model is quantized.

Now let us see the RCC. It has poles whose locations are determined by

$$1 + \frac{1}{8\pi}\gamma^2 = 0$$

$$1 + \frac{1}{8\pi}\gamma^2 \text{ch}^{-2}\gamma\Phi = 0$$

$$\text{ch}\gamma\Phi = 0$$

So only when $\gamma^2 = -\beta^2 < 0$ does the RCC possess poles at

$$\mu_1^2 = 0, \quad \mu_2^2 = m^2 \left(\frac{\beta^2}{8\pi}\right)^{\frac{1}{2(1-\frac{\beta^2}{8\pi})}}$$

As $\beta^2 \rightarrow \beta_c^2, \mu_2^2 \rightarrow \frac{1}{\sqrt{e}}m^2$. Thus we see that there appears another mass scale μ_2^2 in the m. Since the kinks and anti-kinks have masses $M_0 \sim (\frac{m^2}{\beta^2})^{1/(2-\beta^2/4\pi)}$ and the breathers have m $M_n = 2M_0 \sin(n\frac{\pi\beta^2}{16\pi-2\beta^2})$ in the sine-Gordon model[22]-[23], it seems that the mass scale μ_2^2 has nothing to do with the soliton masses.

3.2 D=2 case.

Now the regularized integrals are

$$I_2 = \frac{1}{2\pi}(\mu^2)^{-1/2}$$

$$I_1 = -\frac{1}{2\pi}(\mu^2)^{1/2} + C_1, \quad I_0 = -\frac{1}{6\pi}(\mu^2)^{3/2} + \frac{1}{2}C_1\mu^2 + C_0$$

C_0 and C_1 are two arbitrary constants and only C_1 is nontrivial as in the $D = 1$ case. So we only to fix the mass renormalization condition. Similarly we have $I_{1|\Phi_0} = 0$ and so $C_1 = \frac{1}{2\pi}(m^2 - \mu^2)$. Hence the renormalized coupling constant is

$$\gamma_R^2 = \frac{1 - \frac{\gamma^2}{8\pi}m}{1 + \frac{\gamma^2}{16\pi}m}\gamma^2$$

Similar to eq.(50) we should have

$$\frac{1 - \frac{\gamma^2}{8\pi}m}{1 + \frac{\gamma^2}{16\pi}m} > 0$$

so

$$-\frac{16\pi}{m} < \gamma^2 < \frac{8\pi}{m}$$

From (28) we also have

$$1 + \frac{m}{16\pi}\gamma^2 > 0$$

As in the $D = 1$ case, we have a critical value for β^2 , $\beta_c^2 = \frac{16\pi}{m}$ and $\gamma^2 = \frac{8\pi}{m}$ seems also to be a possible critical point for sinh-Gordon model. As to the low-lying excited states, from the gap equation

$$\frac{\mu^2}{m^2} = \exp[\frac{1}{8\pi}\gamma^2(m - \mu)]\text{ch}\gamma\Phi$$

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we have for $\gamma^2 > 0$,

$$\frac{\mu^2}{m^2} \geq \exp\left[\frac{1}{8\pi}(m - \mu)\right]$$

which means $\mu^2 \geq m^2$, whereas for $\gamma^2 < 0$, we have $\mu^2 \leq m^2$.

The RCC has poles determined by the equations

$$\text{ch}\gamma\Phi = 0$$

$$1 + \frac{1}{16\pi}\mu\gamma^2 = 0$$

$$\left(\frac{\mu^2}{m^2}\right)^2 \exp\left[\frac{1}{4\pi}\gamma^2(\mu - m)\right] + \frac{1}{16\pi}\mu\gamma^2 = 0$$

(where we take $\mu > 0$). So the poles are $\mu_1^2 = 0$, $\mu_2^2 = (\frac{16\pi}{\beta^2})^2$ and μ_3^2 , which is determined by the equation (67). These poles exist only for $\gamma^2 < 0$. Thus after quantization we have two mass scales μ_2^2 and μ_3^2 apart from the mass parameter m at tree level.

3.3 D=3 Case

Now the regularized integrals I_n are

$$I_3 = \frac{1}{6\pi^2\mu^2}, \quad I_2 = -\frac{1}{4\pi^2} \ln \frac{\mu^2}{\mu_s^2}$$

$$I_1 = \frac{1}{8\pi^2}\mu^2 \left(\ln \frac{\mu^2}{\mu_s^2} - 1\right) + C_2$$

$$I_0 = \frac{1}{32\pi^2}\mu^4 \left(\ln \frac{\mu^2}{\mu_s^2} - \frac{3}{2}\right) + \frac{1}{2}C_2\mu^2 + C_3$$

where μ_s^2 , C_2 and C_3 are arbitrary constants and C_3 is trivial. So we need both mass renormalization and coupling constant renormalization. According to the renormalization scheme (48) we have $I_{1|\mu=m} = 0$. So

$$C_2 = \frac{1}{8\pi^2}m^2 \left(1 - \ln \frac{m^2}{\mu_s^2}\right)$$

Therefore from (40) we have

$$\gamma_R^2 = \frac{1 + \frac{1}{16\pi^2}m^2\gamma^2 \ln \frac{m^2}{\mu_s^2}}{1 - \frac{1}{32\pi^2}m^2\gamma^2 \ln \frac{m^2}{\mu_s^2}}\gamma^2$$

To fix μ_s^2 we choose the same scheme as the mass renormalization : *the Φ_0 -renormalized coupling constant equals the coupling constant at tree level: $\gamma_R^2 = \gamma^2$* . So we have $\gamma^2 = 0$ or $\mu_s^2 = m^2$. The first case is trivial and can not determine μ_s^2 . So only the second is of physical significance. we arrive at an important conclusion that *the $D = 3$ sinh(sine) -Gordon model is non-trivial.*

is an important discrepancy between our analysis and that of Ingermanson.

The bounds for the particle mass of the low-lying excited states can also be obtained. From the gap equation and the fact that

$$I_1(\mu^2) = \frac{1}{8\pi^2}\mu^2 \ln \frac{\mu^2}{m^2} + \frac{1}{8\pi^2}(m^2 - \mu^2)$$

we have

$$\ln \frac{\mu^2}{m^2} = \frac{\gamma^2 \mu^2}{32\pi^2} \ln \frac{\mu^2}{m^2} + \frac{\gamma^2 m^2}{32\pi^2} \left(1 - \frac{\mu^2}{m^2}\right) + \ln \text{ch} \gamma \Phi$$

If we define $x \equiv \mu^2/m^2$ and $\kappa \equiv \gamma^2 m^2/(32\pi^2)$, then the gap equation (72) can be written as

$$\ln x = \kappa(x \ln x + 1 - x) + \ln \text{ch} \gamma \Phi$$

. Consider the solution of this equation by graphical means. First when $\gamma^2 > 0$, for $\phi = 0$, the l.h.s. of the l.h.s. will intersect that of the r.h.s. at two points: $x_1 = 1$ and a larger x_2 . As Φ increases, the first root increases and the second one decreases. At some critical Φ_{cri} , the two will meet. For $\Phi > \Phi_{cri}$, there will be no root for $0 < x < \infty$. For $\Phi < \Phi_{cri}$, in order to guarantee a local minimum of \mathcal{E} , the root must satisfy eq.(28), i.e. $\kappa \ln x > 1$ and $I_2 \neq 0$. Therefore, for $\Phi < \Phi_{cri}$, $x = 1$ is not definitely the local minimum. In general we have that when $\mu^2(\Phi) \geq m^2$. For $\gamma^2 > 0$, there is only one root of the gap equation. In this case, if $\beta\Phi = 2n\pi$, the root $x = 1$ is not either a local minimum. Since $\ln \cos \beta\Phi \leq 0$, we have $x \leq 1$. Certainly, eq(28) must be also satisfied at a root if it is a local minimum of \mathcal{E} .

The analysis of RCC is a little more difficult. Eq(44) gives a pole $\mu_1^2 = 0$ when $\gamma^2 < 0$. Eq(46) now reads

$$1 - \frac{\gamma^2 m^2}{32\pi^2} \frac{\mu^2}{m^2} \ln \frac{\mu^2}{m^2} = 0$$

Since for $\gamma^2 > 0$, $\mu^2 \geq m^2$, there is only one solution to it. For $0 < -\gamma^2 < \frac{32\pi^2 e}{m^2}$, $\mu^2 \leq m^2$, there exist two solutions. Eq(46) can also give one pole for the $\gamma^2 > 0$ and two poles for $\gamma^2 < 0$ to take v over a certain range.

4 Summary and Discussion

We have extracted some physical information of sinh(sine)-Gordon model by using the new scheme. We arrive at an important conclusion which is substantially different from Ingermanson's that the $D = 3$ sinh(sine) -Gordon model is non-trivial so long as the regularization constant

is chosen to be m^2 . This should not be surprising because for $D = 3$, the Coulomb gas model can also be transformed to be a sine-Gordon model and there should exist a nontrivial quantum theory for the former. Our conclusions agree exactly with those of Ingermanson for $D = 1, 2$ but disagree for $D = 3$. Furthermore, our analysis shows that for $D = 1, 2$, the RCC has poles for $\gamma^2 < 0$ while the sinh-Gordon model has a critical point γ_c^2 while for $D = 3$, the RCC has poles for both $\gamma^2 > 0$ and $\gamma^2 < 0$. The existence of the poles of the RCC provides some new mass scales as in the sine-Gordon model[11]. Unfortunately we can not still obtain another critical point $\beta_c^2 = 4\pi$ which is almost as important as $\beta_c^2 = 8\pi$ in the $D = 1$ sine-Gordon model[7]. This is perhaps an intrinsic disability of the GEP method.

The poles in RCC reflect the intrinsic properties of the model. They are neither the masses of solitons nor quite the same as the so-called "Landau pole μ_L " like that in QED discussed in previous literatures. In the past, the Landau pole μ_L emerges as an singularity or obstruction on the way of running of cutoff $\Lambda \rightarrow \infty$, or some arbitrary mass scale μ (which stems from some regularization procedure, e.g. the dimensional regularization) approaching to infinity. Of course, there is a similarity between Landau pole and the largest mass scale in our treatment. For example, in ref.[11] it is found that there are three mass scales characterizing the $\lambda\phi^4$ model, among them, the largest one, say μ_c , can only be found by non-perturbative method (like GEP) and evolves into the largest energy scale in the standard model of particle physics where the ϕ -field is coupled to gauge field. At μ_c , the system undergoes a phase transition in vacuum (from symmetry broken phase to symmetric one). We guess that similar phase transition would occur also in the models considered in this paper.

As in the present R-R scheme, there is no explicit divergence (which is substituted by renormalization constants C, μ_s), no counterterm, no bare parameter and no arbitrary running mass scale (all parameters in our treatment are fixed and all running parameters are physical ones) as well. There is no obstruction in the running of cutoff $\Lambda \rightarrow \infty$ and no bare parameter, say γ_0 either, so there is no contradiction in enforcing $\gamma_0 \rightarrow 0$. Hence we claim that there is no "triviality" in $D = 3$ sinh(sine)-Gordon model as that in $\lambda\phi^4$ model[11]. A useful model should be non-trivial. On the other hand, very probably the model has some singularities e.g. some poles of RCC, showing the boundary of its applicability. To go beyond the physics at the singularity is beyond the ability of the QFT under consideration.

As discussed in ref.[9], the QFT is not well-defined by the Lagrangian solely. In GEP scheme

model is defined by the effective Hamiltonian

$$\mathcal{H}_{eff} = \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}(\nabla\Phi)^2 + V_G(\Phi)$$

with V_G containing some arbitrary constants (C, μ_i) . The constant are the necessary complimer the original \mathcal{L} before the model can be well-defined. They are nothing but the values of mass m and coupling constants. In some sense, the renormalization in QFT is just like to reconfirm the ticket before one's departure from the airport. We must keep the same symbol of parametres m , through out the whole calculation.

Once these constants are fixed, the model is well defined and has some prediction power. calculation of eq.(74) at tree level already includes the quantum corrections. We can consider momentum dependent vertices after the first two terms besides V_G in eq.(74) are taken into account. Everything is unambiguous and is well-controlled. The reason why an original "non-renormalizable" model becomes renormalizable in GEP scheme could be understood by an example in quantum mechanics. In the Hamiltonian of hydrogen-like atom, if besides $H_0 = \frac{1}{2\mu^2}\mathbf{p}^2 - \frac{Ze^2}{4\pi r}$, we add a perturbation term, $H' = Ae^{-bp^2} = A\sum_{k=0}^{\infty} b^k p^{2k}$, then the energy correction in eigenstate $|nlm\rangle$ remains finite and fixed to be $\Delta E = \langle nlm | H' | nlm \rangle$ whereas the contribution of individual term in H' , $\langle nlm | p^{2k} | nlm \rangle$, ($k \geq 3$), would *diverge* ! Once again, this example reminds the implication of divergence, which is by no means a very large number. Rather, it is essentially a warning, showing that there might some lack of knowledge or some unsuitability in our treatment. For the moment, we can not claim that what we find is the only finite solution of the model which was believed as non-renormalizable. But we think an outcome from GEP manipulation could be meaningful since the experince in physics often tell us that the nature does not reject the similar possibility.

In the case of $\gamma^2 = -\beta^2 < 0$, i.e. in the sine-Gordon model, the original $V(\phi) \sim \cos\beta\phi$ breaks the discrete translational symmetry: $\phi \rightarrow \phi + \frac{2\pi n}{\beta}$. At first sight, the ansatz of the Gaussian functional Eq.(10) would break this symmetry. First, in general one can not expect that the ground state has the same symmetry as the Hamiltonian [23]. Note that, however, what appears in eq.(10) is the difference $(\phi_x - \Phi_x)$ not ϕ_x itself. Then the contributions of the fluctuations in different configuration of ϕ with $n \neq 0$ are taken into account conceptually for a fixed Φ_x in the path integral. Yet, the contributions for $n \neq 0$ is strongly suppressed. In ref.[24](see also [3]), the soliton line between neighbouring Φ sectors in quantized sine-Gordon model is considered in the $D = 1$ case with

GEP as shown here by eqs.(30),(47) and (53)

$$V_G(\Phi) = \frac{m_R^2}{\beta^2} \left(\frac{\beta^2}{8\pi} - 1 \right) (\cos \beta \Phi)^{8\pi/(8\pi-\beta^2)}$$

which still preserves the symmetry. For the $D = 2$ (or 3) case, through we can not write down explicit GEP like eq.(75) due to the complicate gap equation (63) (or (72)), we are still able to show that the GEP preserves the periodic symmetry,i.e.

$$V_G(\Phi) = V_G\left(\Phi + \frac{2n\pi}{\beta}\right)$$

In summary, the GEP approach combining with the new R-R method does provide a nice computational scheme for non-perturbative QFT.

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